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Solutions of Simultaneous Algebraic Eqns

When there are 'm' linear eqns and 'n' no. of unknowns (variables) it is called simultaneous Algebraic equations.
If $m > n$ then rule of eqns cannot be satisfied.
If $m < n$, the system of linear eqns usually have infinite no. of solution.
We will discuss solution for $m = n$ by using different direct methods and iterative methods.

System of simultaneous linear eqns. is given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n = b_n$$

Criterion for Consistency

1) If $\text{rank}(A) = \text{rank}(A/B) = n$ the system of eqns are consistent & have a unique soln.
2) If $\text{rank}(A) \neq \text{rank}(A/B)$ the system of eqns is inconsistent and hence have no solution.
3) If $\text{rank}(A) = \text{rank}(A/B) < n$, the system of eqn are consistent and have infinite no. of solutions.

We can obtain solution for the above system of equation by using direct methods such as Gauss elimination Method, LU Decomposition Method, Jordan's Method & Crout's Method.

We can also use iterative methods as Jacobi Method, Gauss-Seidel Method & Relaxation Method.

Gauss-Elimination Method

In this method, the variables from the system are successively eliminated & the system is therefore reduced to an upper triangular ~~sysme~~ system. It is described as follows:-

Let the system of eqn be $AX = B$ — ①.

assuming $\det A \neq 0$.

Eqn ① can be expressed as.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ & \;\;\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \text{— ②}$$

First assuming $a_{11} \neq 0$ and divide the first eqn by a_{11} and then we ~~substitute~~ subtract this eqn after multiplied by $a_{21}, a_{31}, \ldots, a_{n1}$ from second, third, ..., n^{th} eqn successively. and get

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\ & \;\;\vdots \\ a'_{n2}x_2 + \cdots + a'_{nn}x_n &= b'_n \end{aligned}\right\} \text{③}$$

Again we divide 2nd eqn of ③ by a'_{22} and then subtract this eqn after multiplied by $a'_{32}, a'_{42}, \ldots, a'_{n2}$ from 3rd, 4th, ..., nth eqn of ③ and get

$$\left.\begin{array}{l} x_1 + a''_{12} x_2 + a''_{13} x_3 + \ldots + a''_{1n} x_n = b''_1 \\ \quad x_2 + a''_{23} x_3 + \ldots + a''_{2n} x_n = b''_2 \\ \quad\quad a''_{33} x_3 + \ldots + a''_{3n} x_n = b''_3 \\ \quad\quad - - - - - - - - \\ \quad\quad a''_{n3} x_3 + \ldots + a''_{nn} x_n = b''_n \end{array}\right\} \quad (4)$$

Similarly continuing in this way we get

$$\left.\begin{array}{l} x_1 + c_{12} x_2 + c_{13} x_3 + \ldots + c_{1n} x_n = d_1 \\ \quad x_2 + c_{23} x_3 + \ldots + c_{2n} x_n = d_2 \\ \quad\quad x_3 + \ldots + c_{3n} x_n = d_3 \\ \quad\quad - - - - - \\ \quad\quad\quad c_{nn} x_n = d_n \end{array}\right\} \quad (5)$$

This is a form of an upper triangular system. From back substitution we can find the solution of the given system of eqns

Example

For a system of equation such as

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \end{array}\right\} — (1)$$

Step I : Eliminate x_1 from 2nd & 3rd eqn. Assuming $a_{11} \neq 0$. Now dividing first eqn by a_{11} & then subtract from 2nd & 3rd after multiplied by a_{21} & a_{31} respectively.

we will get

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + a'_{13}x_3 = b'_1 \\ \quad a'_{22}x_2 + a'_{23}x_3 = b'_2 \\ \quad a'_{32}x_2 + a'_{33}x_3 = b'_3 \end{array}\right\} \text{—(2)}$$

where, $a'_{12} = \dfrac{a_{12}}{a_{11}}$, $a'_{13} = \dfrac{a_{13}}{a_{11}}$, $b'_1 = \dfrac{b}{a_{11}}$

$a'_{22} = a_{22} - a_{21}a'_{12}$

$a'_{23} = a_{23} - a_{21}a'_{13}$; $b'_2 = b_2 - a_{21}b'_1$

$a'_{32} = a_{32} - a_{31}a'_{12}$

$a'_{33} = a_{33} - a_{31}a'_{13}$; $b'_3 = b_3 - a_{31}b'_1$

Step II: Eliminating x_2 from 3^{rd} eqn in (2)

Again assuming $a'_{22} \neq 0$. Dividing eqn (2) by a'_{22} and the subtract from 3^{rd} eqn after multiplied by a'_{32} we get

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + a'_{13} = b'_1 \\ \quad x_2 + a''_{23}x_3 = b''_2 \\ \quad\quad a''_{33}x_3 = b''_3 \end{array}\right\} \text{—(3)}$$

where $a''_{23} = \dfrac{a'_{23}}{a'_{22}}$, $a''_{33} = a'_{33} - a'_{32}a''_{23}$

$b''_2 = \dfrac{b'_2}{a'_{22}}$, $b''_3 = b'_3 - a'_{32}b''_2$

Now evaluating x_1, x_2, x_3 from (3) by back substitution we get the desired solution.

Ques) Solve by Gauss-Elimination Method the following system of equations:-

$$6x + 3y + 2z = 6$$
$$6x + 4y + 3z = 0$$
$$20x + 15y + 12z = 0$$

Soln $\because a_{11} = 6 \neq 0$ we divide the first eqn. by 6 and get

$$x + \frac{1}{2}y + \frac{1}{3}z = 1 \quad \text{——— (1)}$$

Now eliminating x from second and third eqns.

$$\cancel{a'_{12} = a_{12}/a_{11}} \qquad a'_{22} = a_{22} - a_{21}a'_{12}$$

$$a'_{22} = 4 - 6(1/2) = 4 - 3 = 1$$
$$a'_{23} = a_{23} - a_{21}a'_{23}$$
$$= 3 - 6(1/3) = 3 - 2 = 1$$
$$b'_2 = b_2 - a_{21}b'_1$$
$$= 0 - 6(1) = -6$$

Second eqn. is $y + z = -6$ — (2)

$$a'_{32} = a_{32} - a_{31}a'_{12}$$
$$= 15 - 20(1/2) = 15 - 10 = 5$$
$$a'_{33} = a_{33} - a_{31}a'_{13}$$
$$= 12 - 20\left(\frac{1}{3}\right) = \frac{16}{3}$$

$$b'_3 = b_3 - a_{31}b'_1$$
$$= 0 - 20(1) = -20$$

Eqn three will by $5y + \frac{16}{3}z = -20$ — (3)

Now the system of eqn is

$$x + \frac{1}{2}y + \frac{1}{3}z = 1$$

$$y + z = -6.$$

$$5y + \frac{16}{3}z = -20$$

Now eliminating y from eqn (3) with help of eqn (2)

$a'_{22} \neq 0$

$$a''_{23} = \frac{a'_{23}}{a'_{22}} = 1, \quad a''_{33} = a'_{33} - a'_{32}a''_{23}.$$

$$\Rightarrow a''_{33} = \frac{16}{3} - 5(1) = \frac{1}{3}$$

$$b''_2 = \frac{b'_2}{a'_{22}} = \frac{-6}{1} = -6$$

$$b''_3 = b'_3 - a'_{32}b''_2$$

$$= -20 - 5(-6) = -20 + 30 = 10$$

So the required eqn. is

$$\frac{1}{3}z = 10$$

$$\Rightarrow z = 30$$

The system becomes

$$x + \frac{1}{2}y + \frac{1}{3}z = 1$$

$$y + z = -6$$

$$z = 30 \quad \text{— (4)}$$

Using (4) in (2) we get

$$y = -6 - z = -6 - 30 = -36$$

$$\Rightarrow x + \frac{1}{2}(y) + \frac{1}{3}(z) = 1 \Rightarrow x - 18 + 10 = 1$$

$$\Rightarrow x = 9$$

So, $x = 9$, $y = -36$, $z = 30$.

Ans